Fermion Production in the Background of Minkowski Space Classical Solutions in Spontaneously Broken Gauge Theory

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Abstract

We investigate fermion production in the background of Minkowski space solutions to the equations of motion of $SU(2)$ gauge theory spontaneously broken via the Higgs mechanism. First, we attempt to evaluate the topological charge Q of the solutions. We find that for solutions Q is not well-defined as an integral over all space-time. Solutions can profitably be characterized by the (integer-valued) change in Higgs winding number ΔN_H. We show that solutions which dissipate at early and late times and which have nonzero ΔN_H must have at least the sphaleron energy. We show that if we couple a quantized massive chiral fermion to a classical background given by a solution, the number of fermions produced is ΔN_H, and is not related to Q.

I. INTRODUCTION

In this paper we study general properties of classical solutions in $SU(2)$ gauge theory with spontaneous symmetry breaking introduced via the Higgs mechanism. The bosonic sector of the model we consider is that of the standard electroweak theory without the $U(1)$ gauge boson. We work entirely in Minkowski space and look at solutions with finite energy. We add a massive quantized $SU(2)$ doublet chiral fermion, and discuss fermion production in the background of classical gauge and Higgs field solutions. We find that the change in fermion number is not determined by the topological charge, but rather equals the change in the winding of the Higgs field.

In previous work [1], classical solutions in $SU(2)$ gauge theory with no Higgs field were studied. These solutions have the property that in the far past and far future they can be described as spherical shells which propagate without distortion. Furthermore, such solutions have nonzero, non-integer topological charge. In this paper the Higgs field is included and we do not restrict ourselves to the spherical ansatz. As we will see, the solutions we consider here are qualitatively different than those considered in Ref. [1].

The topological charge is defined as

$$Q = \frac{g^2}{32\pi^2} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}) \ . \tag{1.1}$$

The usual argument [2] that leads to integer topological charge requires that the region in space-time where the energy density is nonzero be bounded. For solutions to the equations of motion, energy is conserved and the energy computed on *any* equal time surface is non-vanishing. Thus, for Minkowski space solutions, we have no reason to expect integer values of Q.

The integral in (1.1) is over all of space-time. We can attempt to evaluate it as the limit of a sequence of integrals taken over larger and larger regions of space-time. Q is well-defined if we get the same finite result for (1.1) no matter what sequence of integration regions is chosen so long as in the limit all of space-time is included. We find that Q evaluated on

solutions to the classical equations of motion for spontaneously broken $SU(2)$ gauge theory is not well-defined in this sense. Specifically, we first evaluate (1.1) by doing the integral inside a rectangular space-time box whose size we take to infinity, and obtain a finite result. Then, we redo the calculation using a differently shaped convex box and obtain a *different* finite result. Therefore, the topological charge of solutions to the equations of motion cannot be defined.

Suppose we couple a quantized chiral fermion to the classical gauge and Higgs field backgrounds considered in this paper. The anomaly equation relates Q to $\int d^4x\, \partial_\mu J^\mu(x)$ for an appropriately defined fermion current. However, for solutions to the equations of motion Q is not defined and the implications of the anomaly equation for fermion production are murky. Consider a continuous sequence of configurations (which is not a solution) with the gauge field chosen so that the background has a well-defined nonzero Q and with the Higgs field a nonzero constant. Because the fermion mass is generated by a Yukawa coupling to the Higgs field, the fermion mass is also a nonzero constant. If we make the fermion mass large while keeping the background gauge field fixed, surely no fermionic level will cross the mass gap as we follow the sequence from beginning to end. Thus, in a theory with a massive fermion in a background that does not go from vacuum to vacuum, fermion production cannot simply depend on Q, which is determined only by the gauge field. Since the Yukawa coupling is proportional to the fermion mass, it seems reasonable that the background Higgs field plays a crucial role in fermion level-crossing [5]. In fact, we will argue in Section 5 that the number of fermions produced in a background given by a solution which dissipates at early and late times is equal to the change in the winding number of the Higgs field.

Solutions to the equations of motion are particular examples of continuous sequences of configurations parameterized by t which move through a configuration space described by the gauge field $A_\mu(\mathbf{x})$ and the Higgs field $\Phi(\mathbf{x})$. Vacuum configurations can be characterized by the integer-valued winding number of the gauge field. Sequences of configurations beginning and ending in distinct vacua have a nonzero, integer-valued topological charge given by the difference between the winding number of the gauge field in the final and initial

vacuum configurations. The sphaleron [3] is the lowest energy point on the barrier which a path in configuration space connecting two topologically distinct vacua must surmount. Therefore, vacuum to vacuum sequences of configurations which have $Q \neq 0$ must pass over the sphaleron barrier. However, Q cannot profitably be used to characterize solutions since for solutions Q is not well-defined.

The integer-valued Higgs winding number can be used to characterize even nonvacuum configurations. Let $\varphi = (\varphi_1, \varphi_2)$ be the usual Higgs doublet and define

$$\Phi(\mathbf{x}) \equiv \begin{pmatrix} \varphi_2^* & \varphi_1 \\ -\varphi_1^* & \varphi_2 \end{pmatrix} . \tag{1.2}$$

At points $\mathbf{x}$ where $\rho(\mathbf{x}) \equiv (\varphi_1^*\varphi_1 + \varphi_2^*\varphi_2)^{1/2}$ is non-vanishing, the Higgs field can be associated with a special unitary matrix $U(\mathbf{x})$ through

$$\Phi(\mathbf{x}) = \rho(\mathbf{x})\, U(\mathbf{x}) . \tag{1.3}$$

We consider only those configurations for which the fields approach their vacuum values in the $|\mathbf{x}| \to \infty$ limit. Without loss of generality, we can impose the boundary conditions

$$\lim_{|\mathbf{x}| \to \infty} U(\mathbf{x}) = 1 , \tag{1.4a}$$

$$\lim_{|\mathbf{x}| \to \infty} A_\mu(\mathbf{x}) = 0 , \tag{1.4b}$$

and accordingly only consider gauge transformations which approach unity as $|\mathbf{x}| \to \infty$. Configurations with $\rho \neq 0$ throughout space can be characterized by the Higgs winding number

$$N_H = w[U] \tag{1.5}$$

where the integer-valued winding number of a special unitary matrix U satisfying (1.4a) is

$$w[U] = \frac{1}{24\pi^2} \int \mathrm{d}^3x\, \epsilon^{ijk}\, \mathrm{Tr}\Big(U^\dagger \partial_i U\, U^\dagger \partial_j U\, U^\dagger \partial_k U \Big) . \tag{1.6}$$

N_H is gauge invariant under small gauge transformations, but it changes under large gauge transformations.

Now, we return to our discussion of solutions to the equations of motion, which can be viewed as continuous sequences of configurations parameterized by t. Solutions typically have zeros of ρ isolated in space-time. At times when ρ is everywhere nonvanishing we define $N_H(t)$ as the Higgs winding number of the configuration at time t. Although solutions with nonzero energy do not begin and end in vacua, those which we consider have their energy density approach zero uniformly in space in the $t \to \pm\infty$ limits. We refer to this behavior as dissipation, noting that it is the energy density which dissipates while the energy is conserved and may be large (relative to the sphaleron energy.) In this sense, the solutions we consider begin and end *near* vacua. At very early and late times ρ is approaching its vacuum value and is everywhere nonzero. Therefore, $N_H(t)$ becomes constant in time in the far past and in the far future and we can define

$$\Delta N_H = \lim_{t\to\infty} N_H(t) - \lim_{t\to-\infty} N_H(t) \ . \tag{1.7}$$

Note that this difference is gauge invariant even under large gauge transformations. We will show in Section 4 that if a solution which dissipates at early and late times has $\Delta N_H \neq 0$ then it must begin and end near distinct vacua and must therefore have at least the sphaleron energy.

What do we learn in this paper about fermion production in the background of solutions to the Minkowski space equations of motion which dissipate at early and late times? Such solutions do not have well-defined topological charge; they do have a well-defined change in Higgs winding; it is ΔN_H which counts the number of fermions produced. For a solution to have nonzero ΔN_H, it must have at least the sphaleron energy. This means that nonzero ΔN_H and the associated fermion production cannot occur at finite order in perturbation theory.

The logical next step in our investigation is to consider quantum scattering of massive gauge bosons, rather than the classical field configurations which are the subject of most of this paper. Because we have introduced the Higgs field into the theory, the asymptotic states in the quantum theory are in fact gauge bosons (and not glueballs) and we can therefore begin

to consider quantum scattering. We show that the unexpected behavior of the topological charge of classical solutions (namely that it is not defined) has an analogue in quantum scattering. We attempt to define a quantum operator for which the difference between its expectation value in the initial and final states in some scattering process measures the topological charge associated with that process and find that we cannot do so in a Lorentz invariant fashion.

II. SOLUTIONS TO THE EQUATIONS OF MOTION

In this section, we discuss the behavior of solutions to the classical equations of motion for $SU(2)$ gauge theory spontaneously broken via the Higgs mechanism. The action is

$$S = \int d^4x \left\{ -\frac{1}{2}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + \frac{1}{2}\mathrm{Tr}\,\left(D^\mu\Phi\right)^\dagger D_\mu\Phi - \frac{\lambda}{4}\left(\mathrm{Tr}\Phi^\dagger\Phi - v^2\right)^2 \right\} \;, \tag{2.1}$$

where the 2×2 matrix Φ is related to the Higgs doublet φ by equation (1.3) and where

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \\ D_\mu\Phi &= (\partial_\mu - igA_\mu)\Phi \end{aligned} \tag{2.2}$$

with $A_\mu = A^a_\mu \sigma^a/2$. We use the conventional metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. The action is invariant under the transformation

$$\begin{aligned} A_\mu &\to G A_\mu G^\dagger + \frac{i}{g} G\partial_\mu G^\dagger \equiv A^G_\mu \\ \Phi &\to G\Phi \end{aligned} \tag{2.3}$$

where $G(x) \in SU(2)$. The equations of motion are

$$D_\mu F^{\mu\nu} = \frac{ig}{4}\left[\Phi\left(D^\nu\Phi\right)^\dagger - \left(D^\nu\Phi\right)\Phi^\dagger\right] \;, \tag{2.4a}$$

$$D_\mu D^\mu \Phi = -\lambda\left(\mathrm{Tr}\,\Phi^\dagger\Phi - v^2\right)\Phi \;, \tag{2.4b}$$

where

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig[A_\mu, F^{\mu\nu}]. \tag{2.5}$$

We expect a "typical" solution to dissipate both in the far future and in the far past. By dissipation we mean that at early and late times the energy density approaches zero uniformly throughout space. Not all solutions exhibit dissipation. For example, the sphaleron is a static solution and therefore its energy density is constant in time. One can also imagine solutions which are asymptotically equal to the sphaleron for early (late) times but which dissipate at late (early) times. Thus, by restricting ourselves to solutions which dissipate both in the future and the past we are excluding some solutions from our treatment.

For the solutions we wish to treat, at early and late times the magnitude of the Higgs field is everywhere close to its vacuum value and, in particular, does not vanish. This suggests that we work in unitary gauge, in which U defined in (1.3) is set to unity. However, if $\Delta N_H \neq 0$, then it is impossible to choose a gauge in which $U = 1$ both in the future and in the past. It is, however, possible to choose one gauge in which $U = 1$ in the far past and another gauge in which $U = 1$ in the far future. These two gauges will differ by a large gauge transformation with winding ΔN_H.

At either early or late times we can go into the unitary gauge, in which $U = 1$ and the equations of motion for A_μ and ρ are

$$D_\mu F^{\mu\nu} + \frac{g^2}{2}\rho^2 A^\nu = 0 \ , \tag{2.6a}$$

$$\partial_\mu\partial^\mu\rho - \frac{1}{2}g^2\rho\,\mathrm{Tr}\,(A^\mu A_\mu) + 2\lambda\left(\rho^2 - \frac{v^2}{2}\right)\rho = 0 \ . \tag{2.6b}$$

There is only one vacuum configuration in unitary gauge: $A_\mu = 0$, $\rho = v/\sqrt{2}$. We therefore expand the equations of motion as power series in A_μ and in the shifted field

$$h \equiv \rho - \frac{v}{\sqrt{2}} \ . \tag{2.7}$$

To linear order, the equations of motion are

$$\partial^\mu\left(\partial_\mu A_\nu^{\rm lin} - \partial_\nu A_\mu^{\rm lin}\right) + m^2 A_\nu^{\rm lin} = 0 \ , \tag{2.8a}$$

$$\left(\partial^\mu\partial_\mu + m_h^2\right)h^{\rm lin} = 0 \ , \tag{2.8b}$$

where

$$m = \frac{1}{2} g v \qquad m_h = \sqrt{2\lambda}\, v \ . \tag{2.9}$$

By taking the divergence of (2.8a), we get

$$\partial^\mu A_\mu^{\rm lin} = 0 \ . \tag{2.10}$$

Equation (2.10) is the same equation as the Lorentz gauge condition, which here arises as a consequence of the linearized equations of motion in unitary gauge. Using (2.10), equation (2.8a) becomes

$$\left(\partial_\nu \partial^\nu + m^2\right) A_\mu^{\rm lin} = 0 \ . \tag{2.11}$$

Note that (2.11) are the equations of motion of independent massive vector fields labeled by an $SU(2)$ index but there is no remaining gauge invariance.

The solution to (2.11) takes the form

$$A_\mu^{\rm lin}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k} \left[e^{-ik\cdot x} \epsilon_\mu(k) + e^{ik\cdot x} \epsilon_\mu^*(k) \right] \ , \tag{2.12}$$

where $k^\mu = (\omega_k, \mathbf{k})$ with $\omega_k = (\mathbf{k}^2 + m^2)^{1/2}$, and where the $SU(2)$ valued polarization vector ϵ_μ transforms as a vector under Lorentz transformations and satisfies $\epsilon_\mu k^\mu = 0$. There is a large class of solutions to (2.11) for which $A_\mu^{\rm lin}(\mathbf{x}, t) \to 0$ uniformly in $\mathbf{x}$ as $t \to \pm\infty$. This will certainly be the case if at some time t_0, $A_\mu^{\rm lin}(\mathbf{x}, t_0)$ and $\partial_t A_\mu^{\rm lin}(\mathbf{x}, t_0)$ are sufficiently smooth and vanish for $|\mathbf{x}|$ greater than some R. Under analogous conditions, solutions $h^{\rm lin}$ to (2.8b) also approach zero uniformly in $\mathbf{x}$ at early and late times. Also, the energy density of these solutions to the linearized equations vanishes uniformly in $\mathbf{x}$ as $t \to \pm\infty$.

We now have a picture of the behavior of solutions to the equations of motion which dissipate. We choose to work in a gauge in which $U = 1$ in the far past. In this gauge, A_μ is well approximated at early times by (2.12) with some polarization tensor $\epsilon_\mu^p(k)$. This "past" polarization tensor may be such that as the solution evolves forward in time, energy that was widely separated in space comes together, energy densities grow, and nonlinear

effects become important. If the nonlinearities conspire to prevent the energy density from dissipating in the far future, as for example if a sphaleron is created, then we exclude the solution from our discussion. It is more likely, however, that at late times the energy density is once again spread over a large region of space. There is then a gauge in which A_μ is again well approximated by (2.12) this time with a different polarization tensor $\epsilon^f_\mu(k)$. In the gauge in which $U = 1$ in the far past, however, A_μ in the far future is given by

$$A^f_\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2\omega_k}\, U_f(x) \left[e^{-ik\cdot x} \epsilon^f_\mu(k) + e^{ik\cdot x} \epsilon^{f*}_\mu(k) \right] U^\dagger_f(x) \; + \; \frac{i}{g} U_f(x) \partial_\mu U^\dagger_f(x) \ , \tag{2.13}$$

where $U_f(x)$ is an $SU(2)$ valued gauge function satisfying the boundary condition (1.4a) and where $w[U_f] = \Delta N_H$.

It is possible to go much farther in solving the equations of motion. They can be expanded order by order in g and λ, and the resulting equations can be solved for specified initial data using Greens function methods. This procedure can be organized using tree-level Feynman diagrams. Given $\epsilon^p_\mu(k)$ (and suitable initial data for h) one could solve order by order in g and λ for $\epsilon^f_\mu(k)$. In discussing topological properties of solutions in the rest of this paper we will not need to obtain $\epsilon^f_\mu(k)$ explicitly for a given $\epsilon^p_\mu(k)$, and therefore we will not need to solve the higher order equations of motion.

III. UNDEFINED TOPOLOGICAL CHARGE

The topological charge of a sequence of configurations parameterized by t is defined by

$$Q = \frac{g^2}{32\pi^2} \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta}) \ , \tag{3.1}$$

where the integration is over all space-time. The integrand in (3.1) is a total divergence and Q can be written

$$Q = \int d^4x \, \partial_\mu K^\mu \ , \tag{3.2}$$

where the topological current is

$$K^\mu = \frac{g^2}{32\pi^2} \, \epsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr}(A_\nu \partial_\alpha A_\beta + \frac{2}{3}\, ig \, A_\nu A_\alpha A_\beta) \ . \tag{3.3}$$

For solutions to the equations of motion described in the previous section, we now perform the integration in (3.1) in two different ways and obtain finite, but different, answers.

In our first attempt at evaluating (3.1), we do the integral in a finite space-time box extending from $t = -T$ to $t = T$, then take the spatial extent of the box to infinity, and finally take T to infinity. We call the result of this calculation Q_1. We consider only solutions with compact support, by which we mean that at any time t the fields have vacuum values for $|\mathbf{x}|$ greater than some (t dependent) R. We work in a gauge such that for $|\mathbf{x}| > R$, $A_\mu = 0$ (and $U = 1$) and therefore there is no contribution to (3.2) from the surface at spatial infinity. It is convenient to define the Chern-Simons number

$$N_{CS}(t) = \int d^3x\, K^0(\mathbf{x}, t) \ , \tag{3.4}$$

keeping in mind that N_{CS} is not integer-valued. Then,

$$Q_1 = N_{CS}^f - N_{CS}^p \ , \tag{3.5}$$

where

$$N_{CS}^f = \lim_{T\to\infty} N_{CS}(T) \tag{3.6}$$

and

$$N_{CS}^p = \lim_{T\to-\infty} N_{CS}(T) \ . \tag{3.7}$$

We are considering solutions that have the property that $A_\mu(\mathbf{x}, t) \to 0$ for any fixed $\mathbf{x}$ as $t \to -\infty$, and one might naively conclude that N_{CS}^p is zero. However, care is needed when interchanging time limits and spatial integrals — equation (3.7) requires us to do the d^3x integral at finite time, and only then to take the $T \to -\infty$ limit. For example, the energy density also has the property that it vanishes at any fixed spatial point in the infinite past; however, its spatial integral is constant in time and nonzero.

At early times the solutions we are discussing are well approximated by solutions to the linear equations. We choose the gauge such that in the far past A_μ is of the form (2.12)

with polarization vector ϵ^p. In evaluating the right side of (3.7), we find that only terms quadratic in A_μ which involve products of ϵ^p and ϵ^{p*} are nonzero. All other terms, including the cubic terms in A_μ, vanish by the Riemann-Lebesgue theorem. We obtain

$$N^p_{CS} = \frac{g^2}{32\pi^2}\frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, n_\mu \epsilon^{\mu\nu\alpha\beta}\,\frac{ik_\nu}{\omega_k}\,\mathrm{Tr}\left[\epsilon^{p*}_\alpha(k)\epsilon^p_\beta(k)\right]\ , \tag{3.8a}$$

where $n^\mu \equiv (1,0,0,0)$. The trace is over the $SU(2)$ indices on the polarization vectors. To evaluate N^f_{CS}, we use (2.13) in (3.6), noting that gauge transforming a configuration by a gauge function U adds $w[U]$ to the Chern-Simons number, and obtain

$$N^f_{CS} = \frac{g^2}{32\pi^2}\frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, n_\mu \epsilon^{\mu\nu\alpha\beta}\,\frac{ik_\nu}{\omega_k}\,\mathrm{Tr}\left[\epsilon^{f*}_\alpha(k)\epsilon^f_\beta(k)\right]\,+\,w[U_f]\ , \tag{3.8b}$$

where the polarization vectors $\epsilon^f_\mu(k)$ characterize the solution in the far future. The topological charge given by (3.5) is generically nonzero, finite, and not an integer, and one may be tempted to stop here.

However, we now redo the calculation using a different sequence of integration regions. Whereas above we considered space-time boxes bounded in time by $t = \pm T$, we now consider space-time regions bounded by $t' = -T$ and $t = T$, where t' is a Lorentz transform of t. For concreteness, we take the Lorentz transformation to be a boost of velocity β in the 3-direction, and so have $t' = \gamma(t + \beta x^3)$ with $\gamma = 1/\sqrt{1-\beta^2}$. The surfaces $t = T$ and $t' = -T$ intersect and we take the integration region to be the wedge which includes the origin $\mathbf{x} = 0$, $t = 0$. Note that as T goes to infinity, this wedge includes all of space-time, and we call the result of this calculation Q_2. As before, we do the integral as a surface integral. For solutions with compact support, there is no contribution from the surface at spatial infinity. Also, note that for sufficiently large T the surfaces $t = T$ and $t' = -T$ intersect in a region of space-time where $A_\mu = 0$, and the parts of these surfaces which do not bound the wedge over which we are integrating have $A_\mu = 0$. Therefore, doing the integral in (3.1) over the wedge between $t = T$ and $t' = -T$ which includes the origin and then taking the $T \to \infty$ limit yields

$$Q_2 = N^f_{CS} - {N'}^p_{CS} \tag{3.9}$$

with N^f_{CS} as before and with

$$N'^p_{CS} = \lim_{T\to\infty} \int d^3x' K'^0(\mathbf{x}', -T) \tag{3.10}$$

where x'^μ is the Lorentz transform of x^μ and where K'^μ is the Lorentz transform of K^μ. For (3.10) we obtain an expression identical in form to (3.8a) but with k_μ and ϵ_μ replaced by their Lorentz transforms k'_μ and ϵ'_μ. Thus,

$$N'^p_{CS} = \frac{g^2}{32\pi^2}\frac{1}{(2\pi)^3}\int \frac{d^3k'}{2\omega_{k'}}\, n_\mu \epsilon^{\mu\nu\alpha\beta}\, \frac{ik'_\nu}{\omega_{k'}}\ \mathrm{Tr}\left[\epsilon'^{p*}_\alpha(k')\epsilon'^p_\beta(k')\right] \tag{3.11}$$

(Note that in (3.11) we mean n_μ and not n'_μ.) Using the fact that k_μ and ϵ_μ are Lorentz vectors and that d^3k/ω_k is Lorentz invariant we write

$$N'^p_{CS} = \frac{g^2}{32\pi^2}\frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, \tilde{n}_\mu \epsilon^{\mu\nu\alpha\beta}\, \frac{ik_\nu}{\omega_{k'}}\ \mathrm{Tr}\left[\epsilon^{p*}_\alpha(k)\epsilon^p_\beta(k)\right]\ , \tag{3.12}$$

where $\omega_{k'} = \gamma(\omega_k + \beta k^3)$ and $\tilde{n}^\mu \equiv (\gamma, 0, 0, -\gamma\beta)$. Then, using $\epsilon^p_\mu k^\mu = 0$ to eliminate ϵ^p_0, we find

$$N'^p_{CS} = N^p_{CS} + \frac{g^2}{32\pi^2}\frac{1}{(2\pi)^3}\int \frac{d^3k}{2\omega_k}\, \frac{i\beta m^2}{\omega_k(\omega_k + \beta k^3)} \mathrm{Tr}\ \left[\epsilon^p_1(k)\epsilon^{p*}_2(k) - \epsilon^p_2(k)\epsilon^{p*}_1(k)\right]\ , \tag{3.13}$$

where N^p_{CS} is the previous result (3.8a). We have found $N'^p_{CS} \neq N^p_{CS}$, and therefore $Q_2 \neq Q_1$.

We have now evaluated the integral (3.1) using two different sequences of regions of integration. Once the $T \to \infty$ limit has been taken, Q_1 and Q_2 are integrals of the same integrand over all space-time, and yet they differ. This implies that for solutions with compact support $\int d^4x |\epsilon^{\mu\nu\alpha\beta}\mathrm{Tr}(F_{\mu\nu}F_{\alpha\beta})|$ is infinite, *i.e.* the integral (3.1) is not absolutely convergent. Some integrals which fail to be absolutely convergent can be defined unambiguously by integrating over larger and larger convex regions which grow to encompass all of space-time. (As an example, consider $\int dt\, dx \exp(-x^2)(\sin t)/t.$) Since $Q_1 \neq Q_2$, however, the integral (3.1) cannot be defined in this way and thus fails to be absolutely convergent in a particularly bad way.

The reader may wonder whether the topological charge could be defined by requiring that the integral (3.1) be evaluated between parallel space-like surfaces as for Q_1. To investigate

this, evaluate the integral between $t' = -T$ and $t' = T$, take the $T \to \infty$ limit, and call the result Q_3. Q_3 is given by the difference between ${N'}^p_{CS}$ of (3.13) and the analogous ${N'}^f_{CS}$. Is it possible for Q_3 and Q_1 to be equal for arbitrary Lorentz transformations? For this to occur, it would be necessary that the integral on the right hand side of (3.13) be the same if ϵ^p is replaced by ϵ^f for all values of β. Although ϵ^f is determined by ϵ^p so they cannot be viewed as independent, we know of no conservation law strong enough to guarantee this for all β. Thus we believe that $Q_3 \neq Q_1$.

In summary, the integral (3.1) cannot be defined as an absolutely convergent integral over all space-time and cannot be defined as the limit of the integral over arbitrary sequences of larger and larger convex regions of space-time. We know of no reasonable way to define the topological charge of solutions to the equations of motion.

Note, however, that by setting $m = 0$ in equation (3.13) we see that in the massless theory, Q_1 and Q_2 *are* identical. In fact, it can be shown that in the massless case $\int d^4x |\epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta})|$ is finite and therefore the topological charge is well-defined. This result can be established by using the explicit form of the massless Green's function to show that for solutions with compact support, $\int d^3x |\epsilon^{\mu\nu\alpha\beta} \mathrm{Tr}(F_{\mu\nu} F_{\alpha\beta})|$ falls like $1/|t|^2$ for large $|t|$. Since the topological charge of solutions in the massless theory is well-defined, the explicit calculations done in Ref. [1] are unambiguous.

Note that equation (3.13) can also be seen as a demonstration that the early time Chern-Simons number is not Lorentz invariant in the massive theory, but is Lorentz invariant in the massless theory. (The quantum analogue of this fact is discussed in Section 6.) Given a Lorentz vector K^μ, the standard proof that $\int d^3x\, K^0(x)$ is Lorentz invariant requires that K^μ be conserved, *i.e.* that $\partial_\mu K^\mu = 0$. In this sense, it is actually the Lorentz invariance of $N_{CS}(t)$ at early and late times in the massless theory which is more surprising than the lack of Lorentz invariance of $N_{CS}(t)$ at early and late times in the massive theory, since K^μ is not a conserved current in either case.

IV. ΔN_H AND THE SPHALERON

The sphaleron was introduced [3] in the context of studying sequences of configurations in spontaneously broken $SU(2)$ gauge theory. One way of stating the result of Ref. [3] is that if a continuous sequence of configurations which begins and ends in vacuum has nonzero topological charge, then it must include configurations with at least the sphaleron energy.[1] However, topological charge cannot be used to distinguish solutions to the equations of motion which pass over the sphaleron barrier from those which do not, because solutions do not have well-defined topological charge. As we discussed in the Introduction, solutions can be characterized by the gauge invariant integer ΔN_H, and we show in this section that for solutions which dissipate at early and late times $\Delta N_H \neq 0$ only for solutions which cross the sphaleron barrier and consequently have energies exceeding that of the sphaleron.

Consider configurations specified by $A_i(\mathbf{x})$ and $\Phi(\mathbf{x})$ where we always take $A_0 = 0$. Following Ref. [3], we define a non-negative potential energy functional $V[A_i, \Phi]$ by taking the Hamiltonian associated with (2.1), setting $A_0 = 0$ and dropping all terms involving time derivatives of fields. (Note that configurations are specified by the values of the fields, without reference to their time derivatives.) We obtain

$$V[A_i, \Phi] = \int d^3x \left\{ \frac{1}{2}\mathrm{Tr}\,\left(F_{ij}F_{ij}\right) + \frac{1}{2}\mathrm{Tr}\,\left(D_i\Phi\right)^\dagger D_i\Phi + \frac{\lambda}{4}\left(\mathrm{Tr}\Phi^\dagger\Phi - v^2\right)^2 \right\} , \tag{4.1}$$

which we refer to as the energy of the configuration A_i, Φ.

A continuous sequence of configurations parametrized by t with $N_H(t_1) \neq N_H(t_2)$ can consist entirely of configurations with arbitrarily low energy for t between t_1 and t_2, as we now show. Imagine a sequence along which ρ is everywhere nonzero and along which,

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[1] Any sequence of configurations which begins and ends in distinct vacua has some maximum energy. We assume, as is generally assumed, that this maximum energy cannot be arbitrarily close to zero. It is also generally believed that of all the maximum energy configurations, the one with the lowest energy is the sphaleron of Ref. [3] if λ/g^2 is sufficiently small, while for larger λ/g^2 the deformed sphalerons discovered by Yaffe [6] play this role. What we refer to as the sphaleron is the highest energy configuration on the path of minimum maximum energy, independent of the form of this configuration.

therefore, U is everywhere defined. Since each configuration along the sequence is obtained from preceding configurations by continuous transformations, U varies continuously as t changes, and $N_H(t)$ is constant. To obtain $N_H(t_1) \neq N_H(t_2)$, it is necessary that at some t between t_1 and t_2 there is a point in space at which ρ vanishes (U is undefined) and accordingly the energy density is at least $\lambda v^4/4$. While such a configuration does not have arbitrarily low energy density, it can have arbitrarily low energy. Consider a configuration with $A_i(\mathbf{x}) = 0$ everywhere in space, and with ρ vanishing at one point $\mathbf{x}_0$. Further, suppose that Φ deviates from its vacuum value only in some region of characteristic size L about $\mathbf{x}_0$. Then, the energy of this configuration can be reduced to an arbitrarily small value by reducing L. Thus, a sequence of configurations with $N_H(t_1) \neq N_H(t_2)$ can have arbitrarily low maximum energy.

In the remainder of this section we show that for solutions which dissipate at early and late times, as opposed to more general sequences of configurations, obtaining a different N_H for $t \to \infty$ than for $t \to -\infty$ *does* require at least the sphaleron energy. To this end, we introduce the gradient descent integer N_{GD} which characterizes nonvacuum configurations and which, as explained below, has the property that N_{GD} changes only if the sphaleron barrier is crossed [4].

To define N_{GD} of a configuration, we use that configuration as the $\tau = 0$ initial condition in the equations

$$\partial_\tau A_i = -\frac{\delta V[A_i, \Phi]}{\delta A_i} \tag{4.2a}$$

$$\partial_\tau \Phi = -2\,\frac{\delta V[A_i, \Phi]}{\delta \Phi^*}\ . \tag{4.2b}$$

These equations evolve the initial ($\tau = 0$) configuration in a direction in configuration space which is along the gradient of V with the sign chosen so that V decreases as τ increases. (Equations (4.2) are the equations of motion (2.4) with $A_0 = 0$ and with second order time derivatives replaced by first order τ derivatives.) We will assume that there are no local energy minima in configuration space except the true vacua with $V = 0$. Then, there are two possible outcomes as $\tau \to \infty$. For most configurations, the gradient descent

equations (4.2) evolve the configuration towards a vacuum configuration as $\tau \to \infty$. For such configurations, we define N_{GD} as the Higgs winding number of the vacuum configuration reached from the original configuration as $\tau \to \infty$. (N_{GD} is gauge invariant under small gauge transformations, but it changes under large gauge transformations.) For vacuum configurations, $N_{GD} = N_H$, while for nonvacuum configurations N_{GD} is the winding number of the nearest vacuum, found by sliding down the potential. N_H can change during the slide, and therefore N_{GD} and N_H can differ. There are special configurations for which the gradient descent equations do not lead to a vacuum configuration as $\tau \to \infty$. These configurations mark the boundaries between the basins of attraction of different vacua. For these configurations, N_{GD} is not defined.

A continuous sequence of configurations parameterized by t (not necessarily a solution) which has been put into the $A_0 = 0$ gauge by a t dependent gauge transformation can be characterized by $N_{GD}(t)$, the gradient descent integer of the configuration at time t. Consider a sequence for which the gauge invariant integer $N_{GD}(t_2) - N_{GD}(t_1)$ is nonzero. At some intermediate time there must be a configuration in the sequence with at least the sphaleron energy. This can be seen by constructing the following vacuum to vacuum sequence of configurations: append to the sequence from t_1 to t_2 the two sequences of configurations obtained during the descents from the t_1 and t_2 configurations to their respective vacua. This vacuum to vacuum sequence of configurations connects vacua with different winding numbers. Therefore, the reasoning of Ref. [3] can be applied, and we conclude that for some t between t_1 and t_2, the configuration has energy equal to or greater than the sphaleron energy. Thus, changing N_{GD} requires at least the sphaleron energy.

Let us consider a solution to the Minkowski space classical equations of motion of the kind we discussed in Sections 2 and 3. Since $N_{GD}(t)$ is integer-valued and is typically not constant, there will be times t when it is not defined. However, at very early and late times, when the solution has all fields approaching vacuum values, $N_{GD}(t)$ becomes constant in time and we can define

$$\Delta N_{GD} = \lim_{t\to\infty} N_{GD}(t) - \lim_{t\to-\infty} N_{GD}(t) \ , \tag{4.3}$$

which is gauge invariant even under large gauge transformations. We now show that for such a solution $\Delta N_{GD} = \Delta N_H$. This is equivalent to showing that for sufficiently early and sufficiently late times, when the gradient descent procedure is performed on the configuration, the Higgs winding number does not change during the descent.

First, we gauge transform the solution to $A_0 = 0$ gauge. We use the solution to give the $\tau = 0$ initial configurations in the gradient descent equations

$$\begin{aligned} \partial_\tau A_i &= D_j F_{ji} + \frac{ig}{4}\left[\Phi\left(D_i\Phi\right)^\dagger - \left(D_i\Phi\right)\Phi^\dagger\right] \\ \partial_\tau \Phi &= D_i D_i \Phi - \lambda\left(\mathrm{Tr}\,\Phi^\dagger\Phi - v^2\right)\Phi \end{aligned} \tag{4.4}$$

obtained from the potential energy (4.1). As long as ρ does not vanish, it is convenient to rewrite (4.4) in terms of ρ and U instead of Φ. This will be justified *a posteriori* when we show that for configurations characteristic of solutions at early and late times ρ never vanishes during the descent. It is also convenient to introduce the gauge invariant variable

$$W_i \;\equiv\; \frac{i}{g}\,U^\dagger D_i U = U^\dagger A_i U + \frac{i}{g} U^\dagger \partial_i U = A_i^{U^\dagger}\ . \tag{4.5}$$

Using $\Phi = \rho U$, the definition (4.5), and with the help of

$$\begin{aligned} \mathrm{Tr}\,U^\dagger \partial_\tau U =&\ 0 \\ U^\dagger D_i D_i U =&\ -g^2 W_i W_i - ig\partial_i W_i \end{aligned} \tag{4.6}$$

we write the gradient descent equations (4.4) as

$$U^\dagger\left(\partial_\tau A_i\right)U = D_j^W F_{ji}^W - \frac{g^2}{2}\rho^2\, W_i \tag{4.7a}$$

$$U^\dagger \partial_\tau U = -ig\,\rho^{-2}\partial_i\left(\rho^2 W_i\right) \tag{4.7b}$$

$$\partial_\tau \rho = \partial_i\partial_i\rho - \frac{g^2}{2}\rho\,\mathrm{Tr}\left(W_i W_i\right) - 2\lambda\left(\rho^2 - \frac{v^2}{2}\right)\rho \tag{4.7c}$$

where $D_j^W F_{ji}^W$ is defined as $D_j F_{ji}$ of equations (2.5) and (2.2) with A_i replaced by W_i. (This is a slight abuse of notation since W_i is gauge invariant.) We now use (4.7a) and (4.7b) to obtain

$$\partial_\tau W_i = D_j^W F_{ji}^W - \frac{g^2}{2}\rho^2 W_i - ig\left[\, W_i \,,\, \rho^{-2}\partial_j\left(\rho^2 W_j\right)\,\right] + \partial_i\left\{\rho^{-2}\partial_j\left(\rho^2 W_j\right)\right\} \ . \tag{4.8}$$

We now have gradient descent equations (4.7c) and (4.8) involving only the gauge invariant variables W_i and ρ. We can solve them first, and then use (4.7b) and (4.5) to obtain U and A_i.

We wish to apply the gradient descent procedure to configurations taken from solutions at very early and late times when the energy density is everywhere small. Small energy density means that W_i and $h = \rho - v/\sqrt{2}$ are everywhere small. Therefore, we linearize equations (4.7c) and (4.8) in W_i and h, and obtain

$$\partial_\tau W_i = \left(\partial_j\partial_j - m_W^2\right) W_i \tag{4.9a}$$

$$\partial_\tau h = \left(\partial_j\partial_j - m_h^2\right) h \ . \tag{4.9b}$$

We start at $\tau = 0$ with a configuration in which $|h|$ is everywhere less than some h_0. The solution to (4.9b) valid for all $\tau \geq 0$ is determined by h at $\tau = 0$ and is

$$h(\tau, \mathbf{x}) = \frac{\mathrm{e}^{-m^2\tau}}{(4\pi\tau)^{3/2}} \int d^3\mathbf{y}\, \mathrm{e}^{-(\mathbf{x}-\mathbf{y})^2/4\tau}\, h(0, \mathbf{y}) \ . \tag{4.10}$$

Using (4.10) and noting that $|h(0, \mathbf{y})| \leq h_0$, we find that

$$\begin{aligned} |h(\tau, \mathbf{x})| &\leq \frac{\mathrm{e}^{-m^2\tau}}{(4\pi\tau)^{3/2}} \int d^3\mathbf{y}\, \mathrm{e}^{-(\mathbf{x}-\mathbf{y})^2/4\tau} h_0 \\ &= \mathrm{e}^{-m^2\tau} h_0 \\ &\leq h_0 \ . \end{aligned} \tag{4.11}$$

By starting with a configuration taken from the solution at arbitrarily early or late time, we can make h_0 arbitrarily small and in particular much less than $v/\sqrt{2}$. Because $|h| \leq h_0$ for all $\mathbf{x}$ and for all $\tau \geq 0$, we conclude that the solution to (4.9b) is arbitrarily close to the solution to the full nonlinear descent equations. Therefore, there are no places where $h = -v/\sqrt{2}$ and $\rho = 0$ during the descent. We see that N_H cannot change during the descent.

We have shown that when configurations with arbitrarily small energy density are used as initial conditions for the gradient descent procedure, $N_H = N_{GD}$. This implies that for solutions to the classical equations of motion which dissipate at early and late times, $\Delta N_{GD} = \Delta N_H$, and thus solutions with $\Delta N_H \neq 0$ have at least the sphaleron energy.[2]

Let us recapitulate. For vacuum to vacuum sequences of configurations, $Q = \Delta N_H = \Delta N_{GD}$ is an integer which is nonzero only for sequences which cross the sphaleron barrier. For sequences which do not begin and end in vacuum, the situation is more complicated. Such sequences include solutions, and for solutions Q cannot be defined. A nonzero value for ΔN_{GD} arises only for sequences which begin and end near different vacua and which therefore include configurations with at least the sphaleron energy. However, there are sequences of configurations for which $\Delta N_H \neq 0$ which only include configurations with arbitrarily low energy. For the particular case of solutions which dissipate at early and late times, $\Delta N_H = \Delta N_{GD}$, and such solutions therefore have $\Delta N_H \neq 0$ only if they cross the sphaleron barrier.

V. FERMION PRODUCTION IN A CLASSICAL BACKGROUND

In this section we address the question of fermion production in the background of the classical solutions which we have described. We introduce a quantized fermion field Ψ, and as in the standard electroweak theory, we couple only the left-handed component of the fermion to the non-Abelian gauge field and introduce a Yukawa coupling between the fermion and the Higgs field to give the fermion a gauge invariant mass. The action for the fermion is

$$S^{\text{fermion}} = \int d^4x \bar{\Psi} \left[i\gamma^\mu D_\mu - \frac{\sqrt{2} m_f}{v} \left(\Phi P_R + \Phi^\dagger P_L \right) \right] \Psi \ , \tag{5.1}$$

[2]Our proof that $\Delta N_H = \Delta N_{GD}$ goes through for any continuous sequence of configurations parameterized by t ranging from $-\infty$ to ∞ for which the energy density of the configurations dissipates uniformly in $\mathbf{x}$ as $t \to \pm\infty$. This class of sequences includes the solutions to the equations of motion we consider in this paper, but is more general.

where $D_\mu = \partial_\mu - igA_\mu P_L$, $P_L = \frac{1}{2}(1-\gamma_5)$ and $P_R = \frac{1}{2}(1+\gamma_5)$. Here $A_\mu = A_\mu^a \sigma^a/2$ so the left-handed component of Ψ is an $SU(2)$ doublet. For simplicity, both the up and the down components of Ψ have the same mass m_f. The gauge invariant normal ordered fermion current

$$J^\mu = :\bar{\Psi}\gamma^\mu\Psi: \tag{5.2}$$

is not conserved, that is,

$$\partial_\mu J^\mu = \frac{g^2}{32\pi^2}\,\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,\left(F_{\mu\nu}F_{\alpha\beta}\right)\ . \tag{5.3}$$

As discussed in the Introduction, in a background which does not begin and end in pure gauge, fermion production cannot simply depend on Q. In this section, we show that in the background of a solution whose energy density dissipates for $t \to \pm\infty$, the number of fermions produced is ΔN_H. Our argument applies equally to any background whose energy dissipates uniformly in $\mathbf{x}$ at early and late times, solution or not. Such backgrounds may have well-defined Q, integer or non-integer, or they may have undefined Q. Viewed as an index theorem, our $3+1$ dimensional non-Abelian result generalizes an index theorem of Weinberg [7] for the 2 dimensional Abelian Higgs model.

To begin, consider as a background not a solution but rather a sequence of configurations $A_i(\mathbf{x},t)$ and $\Phi(\mathbf{x},t)$ (with $A_0 = 0$) for $-T \le t \le T$ with the fields pure gauge at $\pm T$, that is

$$A_i(\mathbf{x},-T) = \frac{i}{g}U_p(\mathbf{x})\partial_i U_p^\dagger(\mathbf{x}) \qquad \Phi(\mathbf{x},-T) = \frac{v}{\sqrt{2}}U_p(\mathbf{x}) \tag{5.4a}$$

and

$$A_i(\mathbf{x},T) = \frac{i}{g}U_f(\mathbf{x})\partial_i U_f^\dagger(\mathbf{x}) \qquad \Phi(\mathbf{x},T) = \frac{v}{\sqrt{2}}U_f(\mathbf{x})\ . \tag{5.4b}$$

This means that at $t = \pm T$ the fermion Hamiltonian is gauge equivalent to the free Hamiltonian for a fermion doublet of mass m_f. Also, assume that the background is localized in the sense that there is an R_T such that if $|\mathbf{x}| > R_T$ then $A_i(\mathbf{x},t) = 0$ and $\Phi(\mathbf{x},t) = v/\sqrt{2}$.

In this case the topological charge Q given by (3.1) with the integral over t going from $-T$ to T is an integer

$$Q = w[U_f] - w[U_p] \tag{5.5}$$

which is also the change in Higgs winding ΔN_H between $-T$ and T. Under these circumstances a fermion state which at $t = -T$ has n fermions will evolve into a state with $n + Q = n + \Delta N_H$ fermions at $t = T$. This result is established by a direct application of the work of N. Christ [8] who studied the evolution of the in-vacuum, $|0^{\rm in}\rangle$, in a background which begins and ends in pure gauge. He showed that $|0^{\rm in}\rangle$, which in the far past contains no particles, in the far future is a superposition of states each of which has Q more fermions than anti-fermions. Although Ref. [8] is restricted to massless fermions, the formalism is immediately generalizable to massive fermions as long as the mass terms are gauge invariant as they are in (5.1).

The field theory calculation of fermion production is consistent with the more intuitive level crossing picture. Consider the instantaneous Hamiltonian

$$\mathcal{H}(t) = \gamma^0 \left[-i\gamma^i D_i + \frac{\sqrt{2}}{v}\, m_f \, \left(\Phi P_R + \Phi^\dagger P_L \right) \right] \tag{5.6}$$

which acts on single particle spinor wave functions. For $|\mathbf{x}| > R_T$ we have that $A_i(\mathbf{x}, t) = 0$ and $\Phi(\mathbf{x}, t) = v/\sqrt{2}$ so we can impose periodic spatial boundary conditions on the wave functions. This makes the spectrum of $\mathcal{H}(t)$ discrete. (We further make the spatial box so large that the level spacings are much less than m_f.) The spectral flow $\mathcal{F}$ of $\mathcal{H}(t)$ is defined as the number of eigenvalues of $\mathcal{H}(t)$ which cross zero from below minus the number which cross zero from above as t ranges from $-T$ to T. Given the conditions (5.4) we know that the spectrum of $\mathcal{H}(-T)$, which is the same as the spectrum of $\mathcal{H}(T)$, has a gap between $-m_f$ and m_f. Thus $\mathcal{F}$ can also be viewed as the number of levels which go from below $-m_f$ to above m_f minus the number which go from above m_f to below $-m_f$ as t ranges from $-T$ to T. Now the Atiyah-Patodi-Singer theorem [9] (including the identification of the index with the spectral flow) tells us for the case at hand that $\mathcal{F} = Q$. Thus, the Fock space

calculation which gives that the number of particles produced is $Q = \Delta N_H$ is consistent with the intuitive notion that the number of particles produced is equal to the net number of levels which cross the mass gap.

We now turn to a background which is a solution to the Minkowski space equations of motion which dissipates at early and late times. In this case, Q is not defined. However, we are only interested in solutions which approach pure gauge as $t \to \pm\infty$. Thus at very early and very late times the fermion Hamiltonian is close to the free Hamiltonian and we should be able to make sense of the question of how many fermions are produced.

At early and late times ρ, given by $\Phi(\mathbf{x}, t) = \rho(\mathbf{x}, t)U(\mathbf{x}, t)$, approaches $v/\sqrt{2}$ and since ρ does not vanish U can be defined. At early and late times the gauge invariant field $W_i = A_i^{U^\dagger}$ of (4.5) also approaches zero. More specifically, we can choose T_0 big enough to ensure that

$$\begin{aligned} \left|W_i^a(\mathbf{x}, -T_0)\right| &< \epsilon\, m_f/4g \\ \left|\rho(\mathbf{x}, -T_0) - v/\sqrt{2}\,\right| &< \epsilon\, v/8\sqrt{2} \end{aligned} \tag{5.7a}$$

$$\begin{aligned} \left|W_i^a(\mathbf{x}, T_0)\right| &< \epsilon\, m_f/4g \\ \left|\rho(\mathbf{x}, T_0) - v/\sqrt{2}\,\right| &< \epsilon\, v/8\sqrt{2} \end{aligned} \tag{5.7b}$$

where ϵ is a dimensionless number which can be made arbitrarily small by going to large enough T_0 and where the constants multiplying ϵ have been chosen for later convenience.

Now we pick T_0 so large that conditions (5.7) are satisfied with $\epsilon \ll 1$ and we ask how many fermions are produced between $-T_0$ and T_0. The formalism of Ref. [8] requires that the background start and end in pure gauge. We therefore *construct* a background which agrees with our solution for $-T_0 \le t \le T_0$ but is pure gauge at $t = \pm T$ where $T > T_0$. We want the pure gauge configurations reached at $t = \pm T$ to be close to the configurations of the solution at $t = \pm T_0$. By (5.7), $A_i(\mathbf{x}, \pm T_0) \approx \frac{i}{g} U(\mathbf{x}, \pm T_0)\partial_i U^\dagger(\mathbf{x}, \pm T_0)$ and $\Phi(\mathbf{x}, \pm T_0) \approx \frac{v}{\sqrt{2}} U(\mathbf{x}, \pm T_0)$, so we choose the pure gauge configurations of (5.4) to have $U_p(\mathbf{x}) = U(\mathbf{x}, -T_0)$ and $U_f(\mathbf{x}) = U(\mathbf{x}, T_0)$. The background we construct is

$$\bar{\Phi}(\mathbf{x},t) = \begin{cases} \left[\frac{v}{\sqrt{2}} f(t) + \rho(\mathbf{x},-T_0)\,(1-f(t))\right] U(\mathbf{x},-T_0) & -T \le t \le -T_0 \\ \rho(\mathbf{x},t) U(\mathbf{x},t) & -T_0 \le t \le T_0 \\ \left[\rho(\mathbf{x},T_0)\bar{f}(t) + \frac{v}{\sqrt{2}}\left(1-\bar{f}(t)\right)\right] U(\mathbf{x},T_0) & T_0 \le t \le T \end{cases} \tag{5.8a}$$

$$\bar{A}_i(\mathbf{x},t) = \begin{cases} \frac{i}{g} U(\mathbf{x},-T_0)\partial_i U^\dagger(\mathbf{x},-T_0) f(t) + A_i(\mathbf{x},-T_0)\,(1-f(t)) & -T \le t \le -T_0 \\ A_i(\mathbf{x},t) & -T_0 \le t \le T_0 \\ A_i(\mathbf{x},T_0)\bar{f}(t) + \frac{i}{g} U(\mathbf{x},T_0)\partial_i U^\dagger(\mathbf{x},T_0)\left(1-\bar{f}(t)\right) & T_0 \le t \le T \end{cases} \tag{5.8b}$$

where $f(t)$ goes smoothly and monotonically from 1 to 0 as t goes from $-T$ to $-T_0$ and $\bar{f}(t)$ goes smoothly and monotonically from 1 to 0 as t goes from T_0 to T. For this background the topological charge is $Q = w[U(\mathbf{x},T_0)] - w[U(\mathbf{x},-T_0)]$ which is the change in Higgs winding of the solution between $-T_0$ and T_0. (Recall that T_0 has been chosen so large that $U(\mathbf{x},t)$ does not change its winding for $|t| > T_0$.) The number of fermions produced in the background (5.8) is therefore ΔN_H of the solution.

The background (5.8) only matches the solution for $-T_0 \le t \le T_0$. We now argue that in the background (5.8) no fermions are produced while t is between $-T$ and $-T_0$ and while t is between T_0 and T. For $T_0 \le t \le T$ let us examine the spectrum of the single particle Hamiltonian $\mathcal{H}(t)$ given by (5.6) with the fields $\bar{A}_i$ and $\bar{\Phi}$ of (5.8). Since the spectrum is gauge invariant we can just as well use the fields gauge transformed by $U^\dagger(\mathbf{x},T_0)$, which for $T_0 \le t \le T$ gives

$$\bar{A}_i^{U^\dagger(\mathbf{x},T_0)}(\mathbf{x},t) = \bar{f}(t) W_i(\mathbf{x},T_0) \tag{5.9a}$$

$$U^\dagger(\mathbf{x},T_0)\bar{\Phi}(\mathbf{x},t) = \bar{f}(t)\rho(\mathbf{x},T_0) + \frac{v}{\sqrt{2}}\left(1-\bar{f}(t)\right)\ . \tag{5.9b}$$

The Hamiltonian (5.6) for $T_0 \le t \le T$ now takes the form

$$\mathcal{H}(t) = \mathcal{H}_{\text{free}} + \mathcal{H}'(t) \tag{5.10a}$$

where

$$\mathcal{H}'(t) = \bar{f}(t)\left[-g\,\gamma^0\gamma^i P_L\,W_i(\mathbf{x},T_0) + \frac{\sqrt{2}}{v}\,m_f\,\gamma^0\left(\rho(\mathbf{x},T_0) - \frac{v}{\sqrt{2}}\right)\right] . \tag{5.10b}$$

For any $n \times n$ Hermitian matrix M, the maximum of the absolute value of its eigenvalues, $\|M\|$, is less than or equal to n times the maximum of the modulus of its entries. Now $\mathcal{H}'(t)$ acts on eight component spinors and since $\mathcal{H}'(t)$ is already diagonal in the $\mathbf{x}$ basis we have that

$$\|\mathcal{H}'(t)\| \leq 8\,\bar{f}(t)\,\max_{\mathbf{x}}\left\{\frac{g}{2}\left|W_i^a(\mathbf{x},T_0)\right|\,,\;\frac{\sqrt{2}}{v}\,m_f\left|\rho(\mathbf{x},T_0) - v/\sqrt{2}\right|\right\} . \tag{5.11}$$

Because of (5.7b) we have that

$$\|\mathcal{H}'(t)\| < \bar{f}(t)\,\epsilon\,m_f \leq \epsilon\,m_f\ . \tag{5.12}$$

We can make ϵ arbitrarily small by choosing T_0 sufficiently large. Equation (5.12) implies that for any normalized state $|\psi\rangle$

$$\langle\psi|\mathcal{H}'(t)^2|\psi\rangle < \epsilon^2\,m_f^2 \tag{5.13}$$

whereas for $\mathcal{H}_{\text{free}}$ we have that

$$\langle\psi|\mathcal{H}_{\text{free}}^2|\psi\rangle \geq m_f^2\ . \tag{5.14}$$

Let $|\psi\rangle$ be an eigenstate of $\mathcal{H}(t)$ with eigenvalue E, that is $(\mathcal{H}_{\text{free}} + \mathcal{H}')|\psi\rangle = E|\psi\rangle$. Now if $E = 0$ we have $\mathcal{H}_{\text{free}}|\psi\rangle = -\mathcal{H}'|\psi\rangle$ which by (5.13) and (5.14) is impossible. A simple generalization of this argument shows that if E is an eigenvalue of $\mathcal{H}(t)$ for any t between T_0 and T it must satisfy

$$|E| > m_f - \epsilon\,m_f\ . \tag{5.15}$$

Thus the spectrum of $\mathcal{H}(t)$ has no eigenvalues between $-m_f(1-\epsilon)$ and $m_f(1-\epsilon)$ for all t between T_0 and T and we conclude that no levels can cross zero as t goes from T_0 to T.

We have not developed a field theoretical formalism for discussing particle production in a background which does not go from pure gauge to pure gauge. For the case at hand, however,

with the background (5.8) we are confident that no fermions are produced between T_0 and T because the instantaneous single particle Hamiltonian maintains a gap from $-m_f(1-\epsilon)$ to $m_f(1-\epsilon)$ between T_0 and T. An identical argument leads to the conclusion that no fermions are produced between $-T$ and $-T_0$. Since the number of fermions produced between $-T$ and T is ΔN_H, we have that the number of fermions produced between $-T_0$ and T_0 is ΔN_H. Between $-T_0$ and T_0 the fields in (5.8) are those of our Minkowski space solution. Therefore we conclude that for T_0 arbitrarily large the number of fermions produced in the background of a solution between $-T_0$ and T_0 is ΔN_H. The fact that the topological charge is not defined does not alter this conclusion.

Our result is similar in spirit to that proposed for massless fermions coupled to unbroken $SU(2)$ gauge theory by N. Christ in the concluding section of Ref. [8] and considered recently by Gould and Hsu [10]. At early and late times, solutions have the form (2.12) and (2.13) with $\omega_k = |\mathbf{k}|$. Because we have imposed the boundary condition (1.4b), there is an integer $w[U_f]$ associated with solutions to the equations of motion. Christ proposed that the number of fermions produced in a background of this form is $w[U_f]$. For our solutions in the theory with the Higgs field, $w[U_f] = \Delta N_H$ and we have shown that the number of massive fermions produced is ΔN_H. Our result holds even for fermions with arbitrarily small mass, but our analysis relies on the existence of a gap in the fermion spectrum and cannot be applied to a theory with massless fermions.

We have shown that classical solutions which dissipate at early and late times must have at least the sphaleron energy for the Higgs winding number to change. This means that for solutions with energies below this threshold, no fermions are produced. As we discussed earlier, it is possible to solve the classical equations of motion order by order in perturbation theory in g and λ. This is equivalent to solving them order by order in the amplitudes of the fields. Any process that happens only for solutions with energy above some threshold will never be seen in such a perturbative expansion. Therefore, fermion number violation does not occur in backgrounds obtained by solving the classical equations of motion to any finite order in perturbation theory.

VI. QUANTUM SCATTERING

Until now we have kept the gauge and Higgs fields classical. Here, we make a few remarks on quantum scattering of massive gauge bosons. If we were working with an unbroken $SU(2)$ gauge theory, it would be difficult to proceed from a classical treatment to quantum scattering, since the asymptotic states of the quantum theory are glueballs, which have no classical counterparts. However, in the spontaneously broken theory which we are discussing, the asymptotic states in the quantum theory are the massive quanta of the gauge field itself, whose classical analogues are the solutions to (2.8). We do not attempt a complete quantum mechanical treatment here. The point of this section is only to show that there is a direct quantum analogue of the result of Section 3 that the topological charge of solutions cannot be defined.

Consider the theory whose action is given by (2.1). Using standard diagrammatic methods, one can construct the S-matrix describing the scattering of massive gauge and Higgs bosons order by order in perturbation theory. We do not require that there be many gauge or Higgs bosons in the initial or final states. Because of (3.5), we attempt to construct a Chern-Simons number operator $\hat{N}_{\rm CS}$ and then define the topological charge as the difference between the expectation value of $\hat{N}_{\rm CS}$ in the final and initial states of the scattering process. Because we are only interested in $\langle \hat{N}_{\rm CS} \rangle$ in the initial and final states which consist of massive gauge bosons propagating freely, we can use the free field expansion for the gauge field operator

$$\hat{A}^b_\mu(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{(2\omega_k)^{1/2}} \sum_\lambda \left[e^{-ik\cdot x} \epsilon^b_\mu(k,\lambda) \hat{a}^b_\lambda(k) + e^{ik\cdot x} \epsilon^{b*}_\mu(k,\lambda) \hat{a}^{b\dagger}_\lambda(k) \right] \ , \tag{6.1}$$

where the color index b is not summed over. The sum on λ runs from 1 to 3, the polarization vectors satisfy $\epsilon^b_\mu(k,\lambda) k^\mu = 0$ and $\epsilon^b_\mu(k,\lambda) \epsilon^{b\mu}(k,\lambda') = \delta_{\lambda\lambda'}$, and the creation and annihilation operators satisfy $\left[\hat{a}^b_\lambda(k) \, , \, \hat{a}^{b'\dagger}_{\lambda'}(k') \right] = \delta^3(k-k') \delta_{\lambda\lambda'} \delta_{bb'}$. We now define

$$\hat{N}_{\rm CS} = \frac{g^2}{32\pi^2} \, \epsilon^{\ell m n} \int d^3x \, {\rm Tr} \left[\hat{A}_\ell \partial_m \hat{A}_n + \frac{2}{3} \, i g \hat{A}_\ell \hat{A}_m \hat{A}_n \right] \ . \tag{6.2}$$

Substituting (6.1) in (6.2), normal ordering, and taking either the $t \to \infty$ or the $t \to -\infty$ limit, we obtain

$$\lim_{t\to\pm\infty} \hat{N}_{\rm CS} = \frac{g^2}{32\pi^2} \int d^3k \, \frac{|\mathbf{k}|}{2\omega_k} \sum_b \Big[\hat{n}^b_L(\mathbf{k}) - \hat{n}^b_R(\mathbf{k})\Big] \ , \tag{6.3}$$

where $\hat{n}^b_L(\mathbf{k})$ and $\hat{n}^b_L(\mathbf{k})$ are the number operators for left- and right-handed particles with color index b and momentum k. (For a particle with color label b and with positive momentum in the 3-direction, $\hat{a}^b_L = (\hat{a}^b_1 + i\hat{a}^b_2)/\sqrt{2}$ and $\hat{a}^b_R = (\hat{a}^b_1 - i\hat{a}^b_2)/\sqrt{2}$ and $\hat{n}^b_L = \hat{a}^{b\dagger}_L \hat{a}^b_L$.) In taking the $t \to \pm\infty$ limit of (6.2) to obtain (6.3), any terms which do not have the same number of creation and annihilation operators vanish by the Riemann-Lebesgue theorem. At first glance, it seems that (6.3) can be used to compute the difference between the expectation value of $\hat{N}_{CS}$ in the initial and final states of a scattering process. This result would be interesting, because it is certainly possible to have perturbative (tree-level, in fact) $2 \to 2$ scattering processes in which the difference between the number of left- and right-handed massive gauge bosons changes. In the massless theory $\omega_k = |\mathbf{k}|$ and (6.3) is Lorentz invariant. In the spontaneously broken theory, however, (6.3) is not Lorentz invariant. It is not possible, therefore, to define a Lorentz invariant topological charge as the difference between $\langle \hat{N}_{\rm CS} \rangle$ in the initial and final states of a scattering process.

VII. CONCLUSIONS

Classical backgrounds given by solutions to the equations of motion have undefined topological charge, but they can nevertheless be characterized by the integers ΔN_H and ΔN_{GD}. For general sequences of configurations, ΔN_{GD} is nonzero only if the sphaleron barrier is crossed. This is not true in general for ΔN_H, but for solutions to the equations of motion which dissipate at early and late times, $\Delta N_H = \Delta N_{GD}$. Thus, for dissipative solutions, the Higgs winding number changes only if the energy is greater than the sphaleron energy. When a massive chiral fermion is coupled to such a solution, the number of fermions produced is given by ΔN_H, and is not related to the topological charge.

In previous work in unbroken $SU(2)$ gauge theory [1], it was found that the topological charge of classical solutions can be nonzero at finite order in perturbation theory. This might hint that, in nature, fermion number violating processes could occur at finite order in perturbation theory. Now, with the Higgs field included in the theory, we see how nature can avoid this outcome. First, it is impossible to define the topological charge of classical solutions. Second, while at first glance it seems that topological charge could be generated at finite order in quantum scattering processes, here too it turns out to be undefined. Third, for classical solutions we have shown that the number of fermions produced is not given by the topological charge, but by the change in Higgs winding number. This means that fermion number violation does not occur at finite order in classical perturbation theory. Thus, it seems reasonable to assume that in quantum scattering there is no fermion number violation at any finite order in perturbation theory. This is not a surprising conclusion. What is surprising is that we have arrived here not by finding $Q = 0$, but by finding that Q is not well-defined and does not determine the number of fermions produced.

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